On commuting differential operators and differential subresultants (work in progress)

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Algebraic Methods in Dynamical Systems, Barranquilla,
5-11 October, 2014

Contents

Introduction

Commutative Ordinary Differential Operators.

By J. L. Burchnall and T. W. Chaundy.

(Communicated by A. L. Dixon, F.R.S.—Received December 22, 1926.—Revised February 1, 1928.)

The paper is conveniently divided into two sections. The first contains the general argument and the main propositions unencumbered by proof: in the first paragraph of this section is collected material already published; the succeeding paragraphs of the section are devoted to new results. The second section of the paper includes proofs of these results, together with certain corollaries not essential to the main argument.

Part 1.

I.—*Preamble.*

We make certain notational conventions. There is a single independent variable x; the arbitrary dependent variable of a differential equation or operation is written y. With these exceptions Greek letters denote functions of x and English " lower-case " letters denote constants.

The distinctions extend to symbols of functional form, which will represent polynomials, unless the contrary is stated. Thus

$$f(t) \equiv a_0 t^n + a_1 t^{n-1} + \ldots + a_n$$

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Introduction

Burchnall and Chaundy established a correspondence between commutative pairs of ordinary differential operators and algebraic curves.

With the discovery of *solitons* and the integrability of KdV equation, their theory was applied to nonlinear dynamics, they had discovered the spectral curve.

Algebraic approach to handling the inverse spectral problem for the finite-gap operators, with the spectral data being encoded in the spectral curve and an associated line bundle.

KdV The Schrödinger operator in the stationary case

$$L = -\frac{d^2}{dx^2} + u$$

commutes with

$$P_3 = -\frac{d^3}{dx^3} + \frac{3}{4}\left(u\frac{d}{dx} + \frac{d}{dx}u\right) + \frac{d}{dx}$$

The Lax pair equals

$$KdV_1 = [L, P_3] = LP_3 - P_3L = \frac{1}{4}u_{xxx} - \frac{3}{2}uu_x - u_x$$

In particular, for the Rosen-Morse potential

$$u = \frac{-2}{\cosh^2(x)} \text{ then } [L, P_3] = 0$$

Spectral curve $f(\lambda, \mu) = \mu^2 + \lambda + 2\lambda^2 + \lambda^3$, $f(L, P_3) = 0$.

Differential resultant of 2 ODO's

Defined by Ritt (1932), Berkovich and Tsirulik (1986) and studied by Chardin (1991), Li (1998).

(K, ∂) differential field with a derivation
$P, Q \in K[\partial]$, $\mathrm{ord(P) = n}$, $\mathrm{ord(Q) = m}$

The Sylvester matrix $Syl(P, Q)$ is the coefficient matrix of the extended system

$$\{P, \partial P, \ldots, \partial^{m-1} P, Q, \partial Q, \ldots, \partial^{n-1} Q\}.$$

$Syl(P, Q)$ squared matrix of size $n + m + 1$ and entries in K.

Differential resultant of P and Q,

$$\partial\mathrm{Res}(P, Q) := \det(Syl(P, Q))$$

Example:

$$P = a_2\partial^2 + a_1\partial + a_0, \quad Q = b_3\partial^3 + b_2\partial^2 + b_1\partial + b_0$$

$$\partial\mathrm{Res}(P,Q) =$$

$$\begin{vmatrix} a_2 & a_1 + 2\partial(a_2) & a_0 + 2\partial(a_1) + \partial^2(a_2) & 2\partial(a_0) + \partial^2(a_1) & \partial^2(a_0) \\ 0 & a_2 & a_1 + \partial(a_2) & a_0 + \partial(a_1) & \partial(a_0) \\ 0 & 0 & a_2 & a_1 & a_0 \\ b_3 & b_2 + \partial(b_3) & b_1 + \partial(b_2) & b_0 + \partial(b_1) & \partial(b_0) \\ 0 & b_3 & b_2 & b_1 & b_0 \end{vmatrix}$$

Main properties

- $\partial\mathrm{Res}(P,Q) = AP + BQ$ with $A, B \in K[\partial]$, $\mathrm{ord(A)} < \mathrm{m}$, $\mathrm{ord(B)} < \mathrm{n}$.

$$\partial\mathrm{Res}(P,Q) \in (A,B) \cap K.$$

- $$\partial\mathrm{Res}(P,Q) = 0$$

$$\Updownarrow$$

$P = P_1 R$, $Q = Q_1 R$, with $\mathrm{ord(R)} > 0$, $P_1, Q_1, R \in K[\partial]$.

Poisson formula: Chardin (1991), Previato (1991)

Given monic $P, Q \in K[\partial]$ with respective orders n and m and fundamental systems of solutions $y_1, \dots, y_n$ and $z_1, \dots, z_m$ respectively.

$$\partial\mathrm{Res}(P, Q) = \frac{w(Q(y_1), \dots, Q(y_n))}{w(y_1, \dots, y_n)} = \frac{w(P(z_1), \dots, P(z_m))}{w(z_1, \dots, z_m)}$$
$$= \frac{w(y_1, \dots, y_n, z_1, \dots, z_m)}{w(y_1, \dots, y_n) w(z_1, \dots, z_m)}.$$

with Wronskian

$$w(y_1, \dots, y_n) = \det \begin{bmatrix} y_1 & \cdots & y_n \\ \partial y_1 & \cdots & \partial y_n \\ \vdots & \vdots & \vdots \\ \partial^{n-1} y_1 & \cdots & \partial^{n-1} y_n \end{bmatrix}.$$

Burchnall-Chaundy polynomial

Let C be the field of constants of (K, ∂), C algebraically closed and of characteristic zero.

Burchnall-Chaundy (1928)

Given $P, Q \in K[\partial]$, if $[P, Q] = PQ - QP = 0$ then there exists $f(\lambda, \mu) \in C[\lambda, \mu]$ such that $f(P, Q) = 0$.

Such a polynomial $f(\lambda, \mu)$ is called a Burchnall-Chaundy polynomial.

$$g(\lambda, \mu) = \partial\mathrm{Res}(P - \lambda, Q - \mu) = a_n^m \mu^n - b_m^n \lambda^m + \dots$$

a non trivial polynomial

in $(P - \lambda, Q - \mu) \cap K[\lambda, \mu]$

$g(\lambda, \mu) = A(P - \lambda) + B(Q - \mu)$ with $A, B \in K[\lambda, \mu][\partial]$

if $[P, Q] = 0$ then $g(P, Q) = 0$.

Previato (1991)

Given $P, Q \in K[\partial]$ such that $[P, Q] = 0$ then

$$g(\lambda, \mu) = \partial\mathrm{Res}(P - \lambda, Q - \mu) \in C[\lambda, \mu]$$

and $g(P, Q) = 0$.

$P - \lambda$ and $Q - \mu$ are differential operators with coefficients in the differential field $(K(\lambda, \mu), \partial)$, whose field of constants in the algebraic closure $C_{\lambda,\mu} := \overline{C(\lambda, \mu)}$ of $C(\lambda, \mu)$.

$y_1, \ldots, y_n$ a fundamental system of solutions of $P - \lambda$ over $C_{\lambda,\mu}$, a basis of the $C_{\lambda,\mu}$-vector space $V_{\lambda,\mu} := Ker(P - \lambda)$.

$$W((Q - \mu)(y_1), \ldots, (Q - \mu)(y_n)) = W(y_1, \ldots, y_n)M$$

M is an $n \times n$ matrix with entries in $C_{\lambda,\mu}$.

$$\partial\mathrm{Res}(P-\lambda, Q-\mu) = \frac{w((Q - \mu)(y_1), \ldots, (Q - \mu)(y_n))}{w(y_1, \ldots, y_n)} = \det(M).$$

Lax pair of KdV

$$L = -\frac{d^2}{dx^2} + u(x)$$
$$P_3 = -\frac{d^3}{dx^3} + \frac{3}{4}\left(u(x)\frac{d}{dx} + \frac{d}{dx}u(x)\right) + \frac{d}{dx}$$

$$u = \frac{-2}{\cosh^2(x)} \text{ then } [L, P_3] = 0$$

$$\partial\mathrm{Res}(L-\lambda, P_3-\mu) = -\mu^2 - \lambda(\lambda+1)^2 =$$

$$\begin{bmatrix} -1 & 0 & \frac{-2}{(\cosh(x))^2} - \lambda & 8\,\frac{\sinh(x)}{(\cosh(x))^3} & \frac{4}{(\cosh(x))^2} - 12\,\frac{(\sinh(x))^2}{(\cosh(x))^4} \\ 0 & -1 & 0 & \frac{-2}{(\cosh(x))^2} - \lambda & 4\,\frac{\sinh(x)}{(\cosh(x))^3} \\ 0 & 0 & -1 & 0 & \frac{-2}{(\cosh(x))^2} - \lambda \\ -1 & 0 & \frac{-3}{(\cosh(x))^2} + 1 & 9\,\frac{\sinh(x)}{(\cosh(x))^3} - \mu & \frac{3}{(\cosh(x))^2} - 9\,\frac{(\sinh(x))^2}{(\cosh(x))^4} \\ 0 & -1 & 0 & \frac{-3}{(\cosh(x))^2} + 1 & 3\,\frac{\sinh(x)}{(\cosh(x))^3} - \mu \end{bmatrix}$$

J. Dixmier (1968)

Given $H = \frac{d^2}{dx^2} + x^3 + h$ with $\frac{dh}{dx} = 0$,

$$P = H^2 + 2x$$
$$Q = H^3 + \frac{3}{2}(xH + Hx)$$

is a commutative pair

$$\begin{aligned}\partial\mathrm{Res}(P-\lambda, Q-\mu) &= \det([12 \times 12]) \\ &= (\mu^2 + h - \lambda^3)^2 = f(\lambda,\mu)^2\end{aligned}$$

$$f(P,Q) = P^2 + h - Q^3 = 0$$

G.A. Lathan (1995)

$H = -\frac{d^2}{dx^2} + u(x)$

$$P = H^2$$
$$Q = H^3$$

is a commutative pair and pairs obtained by Darboux transformation

$$\begin{aligned}\partial\mathrm{Res}(P-\lambda, Q-\mu) =& \det([12 \times 12]) \\ =& (\lambda^3-\mu^2)^2 = f(\lambda,\mu)^2\end{aligned}$$

$$f(P,Q) = P^2 - Q^3 = 0$$

Rank of a pair of commuting operators

$P-\lambda, Q-\mu \in K[\lambda,\mu][\partial]$ have no common solution since

$$g(\lambda,\mu) = \partial\mathrm{Res}(P-\lambda, Q-\mu) = a_n^m \mu^n - b_m^n \lambda^m + \ldots \neq 0.$$

f square free part of g
Spectral curve $\Gamma := \{(\lambda,\mu) \in K^2 \mid f(\lambda,\mu) = 0\}$.
$K(\Gamma)$ fraction field of the domain $\frac{K[\lambda,\mu]}{(f(\lambda,\mu))}$

$P-\lambda, Q-\mu \in K(\Gamma)[\partial]$ have common solutions since

$$g(\lambda,\mu) = \partial\mathrm{Res}(P-\lambda, Q-\mu) = 0.$$

The rank of the pair $P-\lambda, Q-\mu$ is the dimension of the space of common solutions, the order of the common nontrivial factor of $P-\lambda$ and $Q-\mu$.

Krichever (1978), the rank of a commutative pair P, Q is

$$r = \gcd(\mathrm{ord(P)}, \mathrm{ord(Q)}).$$

Burchnall and Chaundy (1928)

when their arguments are differential operators.

Now a pair of operators $\phi(\mathrm{D})$, $\psi(\mathrm{D})$ is not in general commutative, but a pair $f(\mathrm{D}), g(\mathrm{D})$, with constant coefficients, is commutative and more generally a pair $f(\mathrm{R}), g(\mathrm{R})$, where R is itself any operator $\phi(\mathrm{D})$. But the class of commutative pairs, is not restricted to the class of pairs $f(\mathrm{R}), g(\mathrm{R})$. This is shown by such an example as

$$\left.\begin{aligned} \mathrm{P} &\equiv x^{-m}\,\delta(\delta-n)(\delta-2n)\ldots(\delta-mn+n) \\ \mathrm{Q} &\equiv x^{-n}\,\delta(\delta-m)(\delta-2m)\ldots(\delta-mn+m) \end{aligned}\right\}, \qquad (1)$$

where $\delta \equiv x\mathrm{D}$. The pair PQ is commutative but not reducible to the form $f(\mathrm{R}), g(\mathrm{R})$, if m, n are interprime.

We here consider the problem of determining the general pair of operators P, Q such that

$$\mathrm{PQ} \equiv \mathrm{QP}.$$

We made an earlier attempt at this problem in a paper* presented to the London Mathematical Society in 1922. Our methods were then inadequate to resolve the general problem, but we were able to establish certain fundamental properties of commutative operators. These we reproduce in outline, as follows :—

Two commutative differential operators P, Q of respective orders m, n satisfy an algebraic identity, with constant coefficients,

$$f(\mathrm{P}, \mathrm{Q}) = 0$$

of partial orders n, m in P, Q respectively. (2)

* ' Proc. London Math. Soc.,' vol. 21, p. 420 (1923).

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$H \in K[\partial]$ of order r
Monic polynomials $p(s)$ and $q(s)$ in $K[s]$ of coprime degrees

$$p(H) = a_0 + a_1 H + \cdots + a_{n-1} H^{n-1} + H^n,$$
$$q(H) = b_0 + b_1 H + \cdots + a_{n-1} H^{m-1} + H^m,$$

of orders nr and mr respectively, thus $r = rank(p(H), q(H))$.

If H commutes with the roots of $p - \lambda$ and the roots of $p - \lambda$ have multiplicity one then

$$\partial\mathrm{Res}(p(H) - \lambda, q(H) - \mu) = \mathrm{Res}(p(s) - \lambda, q(s) - \mu)^r.$$

The polynomial $p(s) - \lambda$ belongs to $\overline{K(\lambda)}[s]$ and factors as

$$p(s) - \lambda = \prod_{i=1}^{n} (s - s_i), \text{ with } s_i \in \overline{K(\lambda)}.$$

Recall $(\overline{K(\lambda)}, \partial)$ is a differential field whose field of constants in $C_1 = \overline{C(\lambda)}$. Let $\{y_i^1, \ldots, y_i^r\}$ be a C_1-basis of $(H - s_i)$. H commutes with s_i implies

$$p(H) - \lambda = \prod_{i=1}^{n} (H - s_i).$$

Thus,

$$\Phi := \cup_{i=1}^{t} \{y_i^1, \ldots, y_i^{\mu_i r}\}$$

is a C_1-basis of $p(H) - \lambda$.

$Hy_i^j = s_i y_i^j$, H and s_i commute, for $\tilde{q} := q(s) - \mu$ imply

$$\partial^k (q(H) - \mu)(y_i^j) = \tilde{q}(s_i)\partial^k y_i^j, \;\; k = 1, \ldots, nr - 1.$$

$$(q(H) - \lambda)(\Phi) := \cup_{i=1}^{t} \{(q(H) - \lambda)(y_i^1), \ldots, (q(H) - \lambda)(y_i^r)\}$$

$$w((q(H) - \lambda)(\Phi)) = \prod_{i=1}^{t} \tilde{q}(s_i)^{\mu_i r} W(\Phi).$$

from the Poisson formula for the differential resultant

$$\partial\mathrm{Res}(p(H)-\lambda, q(H)-\mu) = \prod_{i=1}^{t} \tilde{q}(s_i)^{\mu_i r} = \mathrm{Res}(p(s)-\lambda, q(s)-\mu)^r.$$

Mironov (2012)

Given $H = \frac{d^2}{dx^2} + x^3 + h$ with $\frac{dh}{dx} = 0$,

$$P = H^2 + 6x$$
$$Q = H^5 + \frac{15}{2}(xH^3 + H^3x) + 45(x^2H + Hx^2)$$

is a commutative pair

$$\begin{aligned}\partial\mathrm{Res}(P-\lambda, Q-\mu) &= \det([16 \times 16]) \\ &= (81 + 27h\lambda^2 + \lambda^5 - \mu^2)^2 = f(\lambda,\mu)^2\end{aligned}$$

$$f(P,Q) = P^2 + h - Q^3 = 0$$

Final Remarks

Galoisian study of rank one Burchnall-Chaundy (algebro-geometric or finite gap) operators: Braverman, Etingof, Gaitsgory (1997), Grigorenko (2009), Brehznev.

In particular, Brezhnev made a deep Galoisian study of the finite gap operators related to the KdV hierarchy (2008,2012,2013).

" The integrability of the Direct (spectral) Problems is in some sense equivalent to the integrability of the Inverse Problems"

THANK YOU!